Differentiate (1) with respect to z, denoting z differentiation by primes.

$$V'' = -[ZI' + Z'I] \tag{3}$$

To obtain a differential equation in voltage alone, I may be substituted from (1) and I' from (2). The result is

$$V'' - \left(\frac{Z'}{Z}\right)V' - (ZY)V = 0 \tag{4}$$

Similar procedure, starting with differentiation of (2), yields a second-order differential equation in I:

$$I'' - \left(\frac{Y'}{Y}\right)I'' - (ZY)I = 0 \tag{5}$$

If Z' and Y' are zero, (4) and (5) reduce, as they should, to the equations for a uniform line (Sec. 5.11). When these derivatives are nonzero, representing the nonuniform line discussed, the equations may be solved numerically or with analog equipment for arbitrary variations of Z and Y with distance. A few forms of the variation permit analytic solutions, including the "radial transmission line," where either Z or Y are proportional to z, and their product is constant. Another important case is the "exponential line," which is taken as the example for this article.

Example 5.15
Line with Exponentially Varying Properties

Let us consider a loss-free exponential line with Z and Y varying as follows:

$$Z = j\omega L_0 e^{qz}; \qquad Y = j\omega C_0 e^{-qz} \tag{6}$$

These variations yield constant values of ZY, Z'/Z, and Y'/Y so that (4) and (5) become equations with constant coefficients,

$$V'' - qV' + \omega^2 L_0 C_0 V = 0 \tag{7}$$

$$I'' + qI' + \omega^2 L_0 C_0 I = 0 \tag{8}$$

These have solutions of the exponentially propagating form,

$$V = V_0 e^{-\gamma_1 z}; \qquad I = I_0 e^{-\gamma_2 z} \tag{9}$$

where

$$\gamma_1 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \omega^2 L_0 C_0} \tag{10}$$

$$\gamma_2 = +\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 - \omega^2 L_0 C_0} \tag{11}$$

We see the interesting property of "cutoff" again, for γ_1 and γ_2 are purely real for low frequencies $\omega < \omega_c$ where

$$\omega_c^2 L_0 C_0 = \left(\frac{q}{2}\right)^2 \tag{12}$$

The attenuation represented by these real values, like that for the loss-free filter-type lines, is reactive. This represents no power dissipation but only a continuous reflection of the wave. For $\omega > \omega_c$, however, the values of γ have both real and imaginary parts, which is a different behavior from that of the loss-free filters. Again the real parts represent no power dissipation (see Prob. 5.15b). The values of γ approach purely imaginary values representing phase change only for $\omega \gg \omega_c$.

The greatest use of this type of line is in matching between lines of different characteristic impedance. Unlike the resonant matching sections (Prob. 5.5c), this type of matching is insensitive to frequency. Note the variation of characteristic impedance

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0 e^{-\gamma_1 z}}{I_0 e^{-\gamma_2 z}} = \frac{V_0}{I_0} e^{-(\gamma_1 - \gamma_2)z} = Z_0(0)e^{qz} \tag{13}$$

Thus Z_0 can be changed by an appreciable factor if qz is large enough. The transmission line approximation will become poor, however, if there is too large a change of Z and Y in a wavelength, or in a distance comparable to conductor spacing.

The design of nonuniform matching sections is explored in detail in Ref. 8.

PROBLEMS

5.2a Sketch the function

$$V(z, t) = \cos \omega\left(t + \frac{z}{v}\right) + \frac{1}{2} \cos 2\omega\left(t + \frac{z}{v}\right)$$

versus $\omega z/v$ for values of $\omega t = 0, \pi/2, \pi, 3\pi/2$ and explain how this shows traveling-wave behavior.

5.2b (i) Derive an expression for the characteristic impedance of the parallel-plate line in Fig. 5.2 having a width w and spacing a neglecting the internal inductance of the conductors. Thin-film transmission lines in some computer circuits can be modeled approximately by the parallel-plane line. The line width is usually about 5 μm and the spacing is by means of dielectric of 1 μm thickness and relative permittivity of 2.5 (as is usually true for dielectrics, the relative permeability can be taken as $\approx$1.0).

(ii) Calculate the characteristic impedance Z_0 and wave velocity v.

(iii) Suppose the dielectric thickness is halved and find the new values of Z_0 and v. A better model for such lines is given in Chapter 8.

[8] R. E. Collin, *Foundations for Microwave Engineering*, McGraw-Hill, New York, 1966, Chapter 5.